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Modelling the response of the atmosphere to equatorial forcing

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SUMMARY

The equatorial region is of great importance in understanding the climate, but achieving good tropical performance in climate models is still an outstanding problem. Essential to resolving tropical dynamics is the correct prediction of the steady state behaviour. In this paper a new method is constructed capable of resolving these steady state solutions and hence validating general circulation models (GCMs).

KEY WORDS: semi-geostrophic equations; tropical dynamics; mass transport; rearrangements

1. INTRODUCTION

The time and length scales of atmospheric dynamics range continuously over many orders of magnitude. In this paper the atmosphere is modelled by the time-dependent three-dimensional semi-geostrophic (SG) equations. Whilst the SG equations are an asymptotic approximation to the governing equations of motion on large scales, and therefore appropriate for describing slowly varying synoptic circulations, they do not describe convection which occurs on a much faster timescale. However, the SG equations do admit non-local mass transportation which is an (instantaneous) analogue of these convective processes. This non-local phenomena is induced by the SG stability condition that ensures that solutions evolve such that air parcels are stable with respect to idealized perturbations in their positions. The SG stability condition manifests itself as a constraint on the pressure field and in the tropics prevents large pressure gradient from occurring, restricting solutions to a near steady state. Lagrangian numerical methods that both honour the SG stability constraint and permit non-local advection have been developed [1–3]; however, these approaches are generally incompatible with the fixed

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grids employed in GCMs. The algorithm discussed in this paper translates the benefits of these Lagrangian methods to fixed grid models.

This paper examines the response of the atmosphere to forcing in the equatorial region by varying sea surface temperatures, developing the work of Gill [4]. If, in the tropics, local convective equilibrium together with hydrostatic balance is assumed then much larger horizontal pressure gradients are implied than observed. These pressure gradients are relieved by the ‘Walker’ circulation, the atmospheric response to spatially varying heating. Whilst one might expect such forcing to generate pressure variations near the equator of magnitudes similar to those observed in the extra-tropical regions, such perturbations violate the SG stability condition and instead the heating is reflected in variations in the vertical velocity profile. In contrast GCMs are based on integrating the primitive equation set, relying on a columnwise convective parameterization scheme to diagnose the required vertical mass transport to maintain stability. However, this approach can generate large horizontal pressure gradients which are relieved by transient waves. These waves are artificial, and impede prediction of the real transient waves which are believed to be responsible for tropical variability. In this paper the SG stability constraint is investigated as a means of validating convective parameterization schemes. An efficient stabilization method is crucial for this project as forcing the model with surface heat sources will tend to generate unstable solutions.

2. GOVERNING EQUATIONS

The SG model is considered on an equatorial channel with a Cartesian co-ordinate system (x, y, z), where z is a function of pressure as defined in Reference [5], and the hydrostatic and Boussinesq approximations introduced in Reference [5] have also been employed. The governing equations are then

$$Q\mathbf{u} + \frac{\partial}{\partial t}\nabla s = H, \quad (fv_{\mathrm{g}}, -fu_{\mathrm{g}}, g\theta/\theta_0) = \nabla s, \quad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Gamma \tag{1}$$

where $\partial\Gamma$ is the boundary of the domain, $\mathbf{n}$ is the outward normal,

$$Q = \begin{pmatrix} f\dfrac{\partial v_{\mathrm{g}}}{\partial x} + f^2 & f\dfrac{\partial v_{\mathrm{g}}}{\partial y} & f\dfrac{\partial v_{\mathrm{g}}}{\partial z} \\ -f\dfrac{\partial u_{\mathrm{g}}}{\partial x} & f^2 - f\dfrac{\partial u_{\mathrm{g}}}{\partial y} & -f\dfrac{\partial u_{\mathrm{g}}}{\partial z} \\ \dfrac{g}{\theta_0}\dfrac{\partial\theta}{\partial x} & \dfrac{g}{\theta_0}\dfrac{\partial\theta}{\partial y} & \dfrac{g}{\theta_0}\dfrac{\partial\theta}{\partial z} \end{pmatrix}, \quad H = \begin{pmatrix} f^2 u_{\mathrm{g}} \\ f^2 v_{\mathrm{g}} \\ \dfrac{g}{\theta_0} r \end{pmatrix}, \quad f = 2\Omega \sin \frac{y}{a}$$

The geostrophic velocities are denoted by u_{g} and v_{g}, θ is the potential temperature, $\mathbf{u} = (u, v, w)$ is the advecting wind field and r a heat source. The function s is a geopotential, Ω is the angular velocity of the earth, and a the radius of the earth. Equation set (1) is solved in the equatorial domain Γ, extending $\pm 30^\circ$ north and south, with $0 \leqslant z \leqslant 27\,000\,\mathrm{m}$.

Utilizing the divergence condition on $\mathbf{u}$ and assuming Q is invertible, we obtain a single equation governing the evolution of s, namely

$$\nabla\cdot Q^{-1}\frac{\partial}{\partial t}\nabla s=\nabla\cdot Q^{-1}H,\quad \left(Q^{-1}\frac{\partial}{\partial t}\nabla s\right)\cdot\mathbf{n}=(Q^{-1}H)\cdot\mathbf{n}\ \text{ on }\partial\Gamma \tag{2}$$

However, at the equator f vanishes and hence (2) is degenerate. Therefore, at these points we solve only the z component of the original vector equation $Q\mathbf{u}+\partial_t\nabla s=H$.

Considerable analysis exists regarding the solutions of (1). Firstly, we note that solutions satisfying the SG assumptions are characterized by the eigenvalues of Q being non-negative, corresponding to symmetrically stable, minimum energy atmospheric states [6]. Secondly, we note that Q is closely related to the Hessian of s and hence this stability condition on Q is effectively a constraint on the curvature of the geopotential.

In the special case where the Coriolis force is constant then the SG system can be considered as a conservative transport equation for the potential vorticity, $\det(Q)$, coupled with the inversion of a Monge–Ampére equation to recover s [7]. This formulation both enables a weak existence theory to be constructed and illustrates that the associated optimum transport map minimizes an energy 'cost' and may contain non-local parcel swapping.

In the varying Coriolis force model used here potential vorticity is no longer exactly conserved in a Lagrangian sense. For this model existence of solutions for the shallow water case have been investigated by generalizing the geostrophic co-ordinates of Hoskins [5]. The generalized co-ordinates enable an analogous energy functional to be defined with respect to which solutions are again shown to be a sequence of minimizers [8].

For the constant Coriolis force case geometric integration methods based on the dual Monge–Ampére plus transport formulation have been constructed and can support solutions for which s is non-smooth, characteristic of frontogenesis [1–3]. However, we wish to construct a variable f model on a fixed grid and instead choose to integrate (1) using finite difference techniques, implicitly assuming a smooth solution, and then apply a stability fix at locations violating the non-negativity condition on Q. The stability fix then takes the form of an implicit mass transport scheme. The advantage of this approach is that the initial integration will correspond to advection by a smooth velocity whilst the stabilization step will correspond to non-local mass transportation. These two stages mirror a standard GCM time step in which the mass transport process is modelled by a parameterization scheme. The mass transport implied in the stabilization stage can therefore serve as a powerful validation tool for the parameterization schemes employed in these models.

3. SOLUTION PROCEDURE

The numerical method implemented consists of the following stages:

1. Advect the geostrophic velocities u_g, v_g, and potential temperature θ, by the wind $\mathbf{u}_{\text{total}}$ using a semi-Lagrangian method, thus, enabling these quantities to be approximated at the end of the time step.
2. Evaluate Q at the middle of the time step and check if the stability condition is satisfied, else halve the time step.

3. Solve the elliptic Equation (2) to obtain an approximation to the geopotential s at the end of the time step.
4. Stabilize the solution by computing $s+\varepsilon^{\text{stab}}$ for which the eigenvalues of Q at the end of the time step are all positive at node points within the domain.
5. Compute either the *smooth* wind field comparable to that calculated in a GCM, or the total mass transport

$$\mathbf{u}_{\text{smooth}} = Q^{-1}\left(H - \frac{\partial}{\partial t}\nabla s\right), \quad \mathbf{u}_{\text{total}} = Q^{-1}\left(H - \frac{\partial}{\partial t}\nabla(s+\varepsilon^{\text{stab}})\right) \tag{3}$$

6. Iterate the loop (1)–(5) within the time step until $\mathbf{u}_{\text{total}}$ and $\varepsilon^{\text{stab}}$ are sufficiently converged.
7. Calculate the new geostrophic velocities from $s+\varepsilon^{\text{stab}}$ and continue with the next time step.

This algorithm is implemented using finite differences to approximate the derivatives and a staggered grid to ensure compactness of the various stencils. The pressure correction stage uses a Crank–Nicholson time integration scheme whilst the semi-Lagrangian component utilizes a four-stage Runge–Kutta method. The semi-Lagrangian stages require the advecting wind to be defined throughout the time step and for the first iteration of the above loop this is achieved by extrapolating from the previous three time levels. Having completed the first iteration of this algorithm the predicted velocities from stage (5) are used.

The novel aspect of this work is the stabilization step achieved by minimizing the functional

$$\mathscr{G}(\varepsilon) = \sum_i (\lambda_i(\varepsilon) - \lambda_{\text{tol}})^2 \quad \forall \lambda_i(\varepsilon) < \lambda_{\text{tol}} \tag{4}$$

where $\lambda_i(\varepsilon)$ are the eigenvalues of the Q matrices evaluated using the potential $s+\varepsilon$ at computational nodes within the domain. The aim is to find a stabilizing correction satisfying $\mathscr{G}(\varepsilon^{\text{stab}})=0$. The sum in (4) is taken over all such eigenvalues that are less than the tolerance λ_{tol}, with the exception of the pairs of zero eigenvalues associated with nodes lying on the equator. The minimization is achieved using a gradient descent method with $\lambda_{\text{tol}}=1\times10^{-10}$ and is found to be robust providing that the potential s has not departed too far from a stable state. If the stabilization stage is unsuccessful then the time step could be repeatedly halved.

It is instructive to view the stabilization stage as a rearrangement process in which the method determines the required sources and sinks to re-align the potential s rather than generate a smooth wind with which to advect s by. In this manner the method may have a non-local effect, transferring or rearranging components of the solution from one part of the domain to another, see Reference [3] where this is achieved by a Lagrangian method.

4. NUMERICAL RESULTS AND DISCUSSION

To simplify the analysis of the results a pseudo 2D solution was considered with constant forcing in the longitudinal direction. The solution was approximated on a grid of $3\times11\times5$ cells in the x, y, z directions, respectively. To generate illustrative examples of the

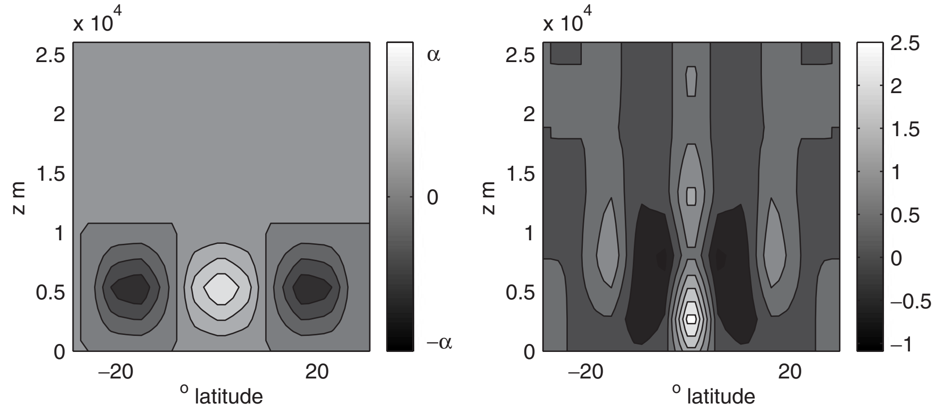

Figure 1. The heat source used as forcing (left) and the mass transport correction (right).

stabilization stage on this coarse grid it was found necessary to consider a larger forcing amplitude than would be encountered globally in a meteorological context. However, the observed phenomena are expected to be found as responses to strong localized forcing for which more moderate circulations would be generated. Initially the atmosphere was considered at rest and, as a relatively coarse discretization has been employed, a reasonably flat vertical potential temperature gradient was specified.

The solution was then initiated by taking three time steps of half an hour each, mildly forced by the profile shown in Figure 1 scaled by $\alpha = 0.001$, followed by a single step with $\alpha = 0.0085$ in order to produce significant instabilities at the end of the time step. A result of this increased forcing is that the computed potential s is unstable, indicated by the presence of the negative eigenvalues of Q in Figure 2 and the inversion of the potential temperature profile in Figure 3.

Applying the stabilization method to this time step renders the eigenvalues of Q positive as indicated by Figure 2. Moreover, we note that the distribution of the maximum eigenvalues is largely undisturbed by the stabilization process indicating that the corrected potential is still close to the unstable potential. The action of the stabilization step is most clearly illustrated at the equator where the single non-zero eigenvalue asserts that the potential temperature must be an increasing function of z. The stabilization of a profile of this nature is shown in Figure 3. At the equator the stabilization is achieved by transport in the vertical direction; however, in the rest of the domain 'slantwise convection' will occur with the potential s attaining a stable configuration through rearrangements not necessarily aligned with the co-ordinate axis.

More generally the effect of the stabilization stage can be assessed by considering the specific mass transport correction evaluated as $\nabla \cdot Q^{-1} \partial / \partial t \nabla \varepsilon^{\text{stab}}$ which indicates the sources and sinks added by this step. For this example the mass transport correction is shown in Figure 1 and we note that the action of this term is largely to redistribute the solution in the unstable zone, where the solution exhibits singular, or near singular, structure. In contrast, the stabilization stage has minimal effect in the smooth regions of the solution where

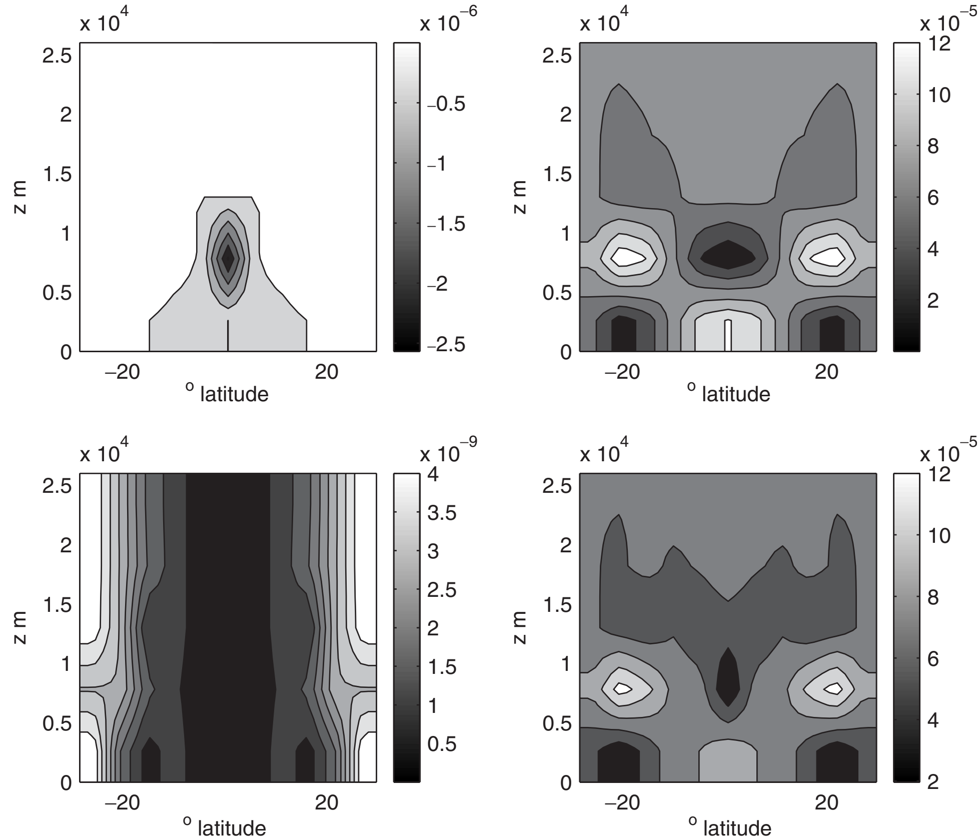

Figure 2. Minimum and maximum eigenvalues of Q pre-stabilization (above) and post-stabilization (below).

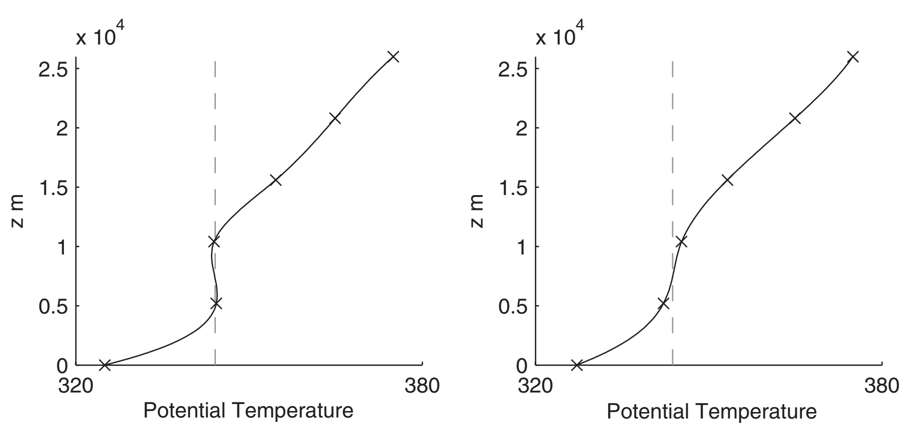

Figure 3. The unstable and stabilized potential temperature profile at the equator.

the finite difference approximations are valid. Critically it is this corrective mass transport that is represented by parameterization schemes in GCMs and deficiency in estimating this transport effect may impair the ability of these models to accurately capture tropical steady state circulations.

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